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On moments based Padé approximations of ruin probabilities

F. Avram[a,*], D.F. Chedom[a,c], A. Horváth[b]

[a] Laboratoire de Mathématiques Appliquées, Université de Pau et des Pays de l'Adour, Avenue de l'Université, BP 1115 64013 Pau, France

[b] Dipartimento di Informatica, Università di Torino, Corso Svizzera 185, 10149, Torino, Italy

[c] Department of Computer Science, University of Yaoundé I, BP 812 Yaoundé, Cameroon

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ABSTRACT

In this paper, we investigate the quality of the moments based Padé approximation of ultimate ruin probabilities by exponential mixtures. We present several numerical examples illustrating the quick convergence of the method in the case of Gamma processes. While this is not surprising in the completely monotone case (which holds when the shape parameter is less than 1), it is more so in the opposite case, for which we improve even further the performance by a fix-up which may be of special importance due to its potential use in the four moments Gamma approximation.

We also review the connection of the exponential mixtures approximation to Padé approximation, orthogonal polynomials, and Gaussian quadrature. These connections may turn out useful for providing rates of convergence.

1. Introduction

The ruin problem for the Cramér Lundberg risk model. Let us recall the classical Cramér–Lundberg model:

$$X_t = u + ct - \sum_{k=1}^{N_t} C_k, \tag{1}$$

used in collective risk theory to describe the surplus $X = \{X_t,\ t \geq 0\}$ of an insurance company. Here, u is the initial capital, ct represents the premium income up to time t, C_k are i.i.d. positive random variables representing the claims made, with cumulative distribution function (cdf) and probability density function (pdf) denoted by $B(x)$ and $b(x)$, and (some) moments denoted by $m_i,\ i = 1, 2, \ldots,$ and $N = \{N_t,\ t \geq 0\}$ is an independent Poisson process with intensity λ modeling the times at which the claims occur.

Let T be the first passage time of the stochastic process $X(t)$ below 0:

$$T := \inf\{t \geq 0 : X(t) < 0\}.$$

The objects of interest in ruin theory are the "finite-time" and "ultimate" ruin probabilities

$$\psi(t, u) = P_u[T \leq t], \qquad \psi(u) = P_u[T < \infty].$$

The problem of approximating ultimate ruin probabilities $\psi(u)$ for the Cramér Lundberg model (1) using data on the distribution $B(u)$ of the claims is a classic of applied probability, dating back to the early 1900s.

* Corresponding author. Tel.: +33 0 5 59 40 75 63; fax: +33 0 5 59 40 75 55.
E-mail addresses: florin.avram@univ-pau.fr, florin.avram@orange.fr (F. Avram), donatien.chedomfotso@univ-pau.fr (D.F. Chedom), horvath@di.unito.it (A. Horváth).

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The Pollaczek–Khinchine formula for the Laplace transform. The relation between $B(u)$ and $\psi(u)$ becomes simpler in the Laplace domain, where the Pollaczek–Khinchine formula yields an explicit expression for the Laplace transform

$$\psi^*(s) = \int_0^\infty e^{-su}\psi(u)du = \frac{1}{s} - \frac{1-\rho}{s(1-\rho b_e^*(s))},$$

where $\rho = \lambda m_1/c$, and $b_e(x) := m_1\bar{B}(x)$, $b_e^*(s) = m_1 s(1-b^*(s))$ denote the stationary excess claim distribution and its Laplace transform — see [1].[1]

Laplace inversion. In general, assuming complete knowledge of the claims distribution, recovering $\psi(u)$ is a problem of Laplace transform inversion, and as such, it could be attacked numerically via the current off-shelf inversion programs available. However, for non-rational symbols, like, for example, log-normal claims, Laplace inversion may involve the non-trivial task of numerical integration of a possibly highly oscillating function.

Claims with rational Laplace transforms. In the case of light tailed claims with rational Laplace transform

$$\bar{B}^*(z) = \frac{\sum_{k=0}^{K-1} a_k z^k}{z^K + \sum_{k=0}^{K-1} b_k z^k}, \tag{2}$$

the Pollaczek–Khinchin formula followed by partial fractions and Laplace inversion yields immediately the eventual ruin probabilities.

Furthermore, the distribution may also be expressed in "matrix exponential" form

$$\bar{B}(x) = \boldsymbol{\beta} e^{\boldsymbol{B}x}\mathbf{1}, \qquad \Leftrightarrow \bar{B}^*(s) = \boldsymbol{\beta}(s\boldsymbol{I}-\boldsymbol{B})^{-1}\mathbf{1} \tag{3}$$

with B a matrix of order K, see for example [2]. This representation renders Laplace inversion unnecessary, and the ruin probability is "explicit" (see [1]):

$$\psi(u) = \boldsymbol{\eta} e^{\mathbf{Q}u}\mathbf{1} \tag{4}$$

where $\mathbf{Q} = \boldsymbol{B} + (-\boldsymbol{B})\mathbf{1}\boldsymbol{\eta}$, $\boldsymbol{\eta} = \rho\boldsymbol{\beta}(-\boldsymbol{B})^{-1}$, $\rho = \lambda/c$.

Exponential mixtures. Our goal is to approximate a given density $f(t)$ (of the claims) by a sum of exponentials

$$f(t) \sim \sum_{i=1}^K w_i\alpha_i e^{-\alpha_i t}, \tag{5}$$

which has $2K-1$ free parameters as we must have $\sum_{i=1}^K w_i = 1$.

The choice of a "best exponential mixture approximation" is not obvious. As far as simplicity, one favourite is approximation by the method of moments.

The "extended Vandermonde" system. The method of moments applied to (5) yields

$$\sum_{i=1}^K w_i p_i^k = c_k, \quad \text{for } k = 0, 1, 2, \ldots, 2K-1 \tag{6}$$

where c_k denotes the normalised moments $m_k/k!$ of the rv whose density is to be approximated and $p_k = \alpha_k^{-1}$.

The resulting "extended Vandermonde system":

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ p_1 & p_2 & \ldots & p_K \\ p_1^2 & p_2^2 & \ldots & p_K^2 \\ \ldots & \ldots & \ldots & \ldots \\ p_1^{2K-1} & p_2^{2K-1} & \ldots & p_K^{2K-1} \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_K \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{2K-1} \end{pmatrix} \tag{7}$$

is linear in the parameters w_k, but non-linear in p_k, $k = 1, 2, \ldots, K$.

Note that the first $2K-2$ equations of the system (7) may be written in matrix form as

$$H_K = V_K(p, K)\, Diag(w_{k=1,\ldots,K})\, V_K(p, K)^t,$$

[1] Note that the Pollaczek–Khinchine formula yields equally the stationary distribution of the waiting time in the M/G/1 queue, and in fact the distribution of any geometric compound sums. Furthermore, it has a straightforward generalisation to the distribution $\psi(u) = P[\overline{Y} > u]$, where $\overline{Y}$ is the maximum of a spectrally positive Levy process Y with negative drift.

where

$$H_K = \begin{pmatrix} c_0 & c_1 & \cdots & c_{K-1} \\ c_1 & \dots & c_{K-1} & c_K \\ \vdots & \dots & \dots & \dots \\ c_{K-1} & c_K & \cdots & c_{2K-2} \end{pmatrix}$$

$$V_K(p, K) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ p_1 & p_2 & \dots & p_K \\ p_1^2 & p_2^2 & \dots & p_K^2 \\ \dots & \dots & \dots & \dots \\ p_1^{K-1} & p_2^{K-1} & \dots & p_K^{K-1} \end{pmatrix}. \tag{8}$$

This is known as the Hankel matrix factorisation, see for example [3].

The Hermite/Padé approximation. Instead of (5), one may consider approximating the moment generating function $f^*(s) = E e^{sC}$ by a **rational function**:

$$f^*(s) \approx \frac{a_{K-1}(s)}{b_K(s)} = \frac{a_0 + \cdots + a_{K-1}s^{K-1}}{b_0 + b_1 s + \cdots + b_{K-1}s^{K-1} + s^K} = \sum_{k=0}^{2K-1} c_k s^k + \cdots \tag{9}$$

where $c_k = m_k/k!$.

Finding the "Laplace parameters" a_i, b_i in (9) requires only solving a linear system of $2K$ equations

$$a_i = \sum_{j=0}^{i} b_{i-j} c_j, \quad i = 0, \dots, 2K-1 \tag{10}$$

where $a_i = 0$ if $i > K-1$, $b_K = 1$ and $b_i = 0$ if $i > K$. The first K equations yield the a_i in terms of the b_i, and the next K equations form a Hankel system

$$\begin{pmatrix} c_0 & c_1 & \cdots & c_{K-1} & c_K \\ c_1 & \dots & c_{K-1} & c_K & c_{K+1} \\ \vdots & \ddots & \ddots & \dots & \vdots \\ c_{K-1} & c_K & \cdots & c_{2K-2} & c_{2K-1} \end{pmatrix} \times \begin{pmatrix} b_K \\ b_{K-1} \\ \vdots \\ b_1 \\ b_0 \end{pmatrix} = 0. \tag{11}$$

Altogether, we get explicit expressions of a_i, b_i in terms of the coefficients c_k of the series expansions of the moment generating function (for a recursive approach, one may also use the continued fraction representation of (9)).

Thus, the non-linear Vandermonde moments equations (7) may be replaced by a Hankel linear system (11) for the coefficients of an associated polynomial $b_K(x)$ (9).

The "Stieltjes/Bernstein/Krein" representation. The approximation (5) is most natural when our density may be represented as

$$f(x) = \int_A \alpha e^{-x\alpha} \mu(d\alpha). \tag{12}$$

Note that in this case the normalised moments $m_k/k!$ of $F(x)$ coincide with the negative moments of the "Stieltjes/Bernstein/Krein" representing measure:

$$\begin{aligned} \frac{\int_A x^k F(dx)}{k!} &= \frac{\int_A x^k \int_A e^{-x\alpha} \alpha \mu(d\alpha) dx}{k!} = \frac{\int_A \left(\int_A x^k \alpha e^{-x\alpha} dx\right) \mu(d\alpha)}{k!} \\ &= \int_A \alpha^{-k} \mu(d\alpha). \end{aligned} \tag{13}$$

Changing variables $p = 1/\alpha$, we arrive at the conclusion.

Proposition 1. *Suppose the distribution to be approximated admits a "Stieltjes/Bernstein/Krein" representation*

$$\bar{F}(x) = \int_A e^{-x/p} \nu(dp), \tag{14}$$

and that the moments

$$c_i = \int_A s^i \nu(ds), \quad i \le 2K-1$$

exist (in which case c_i coincides with the normalised moments of the claims).

Then the Hankel system (11) is equivalent to the orthogonality relations of the polynomial $x^K b_K(x^{-1})$ to lower powers, with respect to the representing measure $\nu(\mathrm{d}x)$, or, equivalently, to the orthogonality of $b_K(x)$ to lower powers, with respect to the representing measure $\mu(\mathrm{d}x)$ defined in (12).

The roots of $b_K(x)$ are the nodes of a K-node Gaussian quadrature with respect to the measure $\mu(\mathrm{d}x)$.

Proof. Let $\tilde{b}_K(x) = x^K b_K(x^{-1}) = \sum_{j=0}^{K} \tilde{b}_j x^j$ where $\tilde{b}_j = b_{K-j}$. Writing the Hankel equations (10) for $i = K + k,\ k = 0, \ldots, K-1$, in terms of $\tilde{b}_j$, we get:

$$\sum_{i=0}^{K} b_i c_{K+k-i} = \int_A \sum_{j=0}^{K} \tilde{b}_j s^{j+k} \nu(\mathrm{d}s) = \int_A \tilde{b}_K(s) s^k \nu(\mathrm{d}s) = 0 \quad \text{for } k = 0, \ldots, K-1. \tag{15}$$

The orthogonality relations of the polynomial $b_K(x)$, with respect to the representing measure $\mu(\mathrm{d}x)$, are established similarly, and the connection to Gaussian quadrature is well known, and its history is reviewed in Section 2. □

Note. For results on the convergence as $K \to \infty$, see [4].

The advantage of moments based methods. It may be argued that the only reliable information available in insurance data is anyway contained in the first few moments of the claims and interarrival time distributions, which explains the importance of non-parametric moments based approximations, like the one implemented below.

The fact that the coefficients of mixture models may be obtained from the moments by two linear systems of Hankel and Vandermonde structure – see Section 2 – which may be solved numerically efficiently is another bonus of the method of exponential mixtures.

Related approaches. Several other approaches have been proposed for the construction of exponential mixtures. Many of these construct a hyperexponential distribution, i.e., a mixture of exponential distributions which is in the form of (5) but where all w_i are positive, resulting in a monotone decreasing pdf. Consequently, these methods use only a subset of the distributions used in this paper. A typical approach to build a hyperexponential distribution is provided by the principle of maximum likelihood [5,6] which result in an iterative procedure. Depending on the number of parameters, these methods can suffer from slow convergence. A much faster, heuristic approach was provided in [7], but note that this method was designed to approximate distributions with heavy tails, and that our method, which is essentially exact for exponential mixtures, will clearly outperform it. An interesting procedure, based on the Jacobi polynomial expansion, was proposed in [8]. This approach goes beyond hyperexponential distributions and can provide non-monotone decreasing pdfs but uses only a subset of the form (5), because the pole structure of the resulting approximation is fixed. A drawback of this approach is the necessity of setting four parameters which requires a non-trivial trial and error phase. The reader is invited to experiment himself with these two approaches using the programs at http://www.di.unito.it/~horvath/additional/.

Contents. This paper is a contribution to the theory of Laplace inversion, from an actuarial point of view. We replaced the method of moments for exponential mixtures with an **arbitrary number of moments** in its historical context, as an application of the Hermite/Padé approximation, and reviewed in Proposition 1 its connections to orthogonal polynomials and Gaussian quadrature. This connection may turn out useful for providing error bounds, the scarceness of which is a current weakness of ruin theory.

The method of moments has its natural limitations, noticed already in Example 3 in the case of Gamma claims with $\alpha > 1$ non-integer. We provided therefore a fix-up in this case – see Example 4 – which is of special importance due to its potential use in the four moments Gamma approximation — see Remark 1, as well as its potential use as a conjugate prior in Bayesian approaches.

The paper is organised as follows. In Section 2 we provide some historical background of approximation theory. In Section 3 we describe our approximation procedure, and review issues regarding positivity of the resulting measure (Section 4). Finally, in Section 5 we present numerical results.

2. The solution of the extended Vandermonde system

The solution of (7) is a classic: see for example [9–12] or [13], Thm 2.1. This system was probably first encountered in [14] in the problem of fitting a curve by combinations of exponentials, see [15]. The system (7) was later encountered in [16] in the problem of choosing nodes and weights for a "quadrature" rule

$$\int_A g(x)\nu(\mathrm{d}x) \sim \sum_k w_k g(p_k), \quad A \in \mathbb{R}$$

which will be exact for polynomials $g(x)$ of a degree as large as possible. Gauss considered $\nu(\mathrm{d}x) = \mathrm{d}x$, in which case (7) holds with $c_k = \int_A x^k \mathrm{d}x$.

Later, [17] noticed the key fact that when p_k are Gauss's nodes, then the polynomial

$$b_K(s) = \sum_{k=0}^{K} b_k s^k = b_K(s-p_1)(s-p_2)\ldots(s-p_K) \tag{16}$$

is orthogonal over A to all polynomials of degree less than K, and his coefficients b_k satisfy the linear Hankel system:

$$\begin{pmatrix} c_0 & c_1 & \cdots & c_{K-1} & c_K \\ c_1 & \cdots & c_{K-1} & c_K & c_{K+1} \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ c_{K-1} & c_K & \cdots & c_{2K-2} & c_{2K-1} \end{pmatrix} \times \begin{pmatrix} b_K \\ b_{K-1} \\ \vdots \\ b_1 \\ b_0 \end{pmatrix} = 0$$

where $c_i = \int_A s^i \nu(\mathrm{d}s)$. Later, [18] generalised Gauss and Jacobi's results to quadrature rules for general measures $\int_A \nu(\mathrm{d}x)$ — for a recent exposition, see [4].

The Hankel system (11) may also be obtained directly from the extended Vandermonde system (7) (see [15]). Indeed, letting b_i denote the coefficients of a polynomial with roots p_j, note that for $k = 0, \ldots, K-1$ it holds by (7) that $c_{i+k} = \sum_{j=1}^K w_j p_j^{k+i},\ i = 0, \ldots, K$. Therefore:

$$\sum_{i=0}^K b_i c_{i+k} = \sum_{i=0}^K b_i \left(\sum_{j=1}^K w_j p_j^{k+i} \right) = \sum_{j=1}^K \sum_{i=0}^K b_i w_j p_j^k p_j^i$$
$$= \sum_{j=1}^K w_j p_j^k \left(\sum_{i=0}^K b_i p_j^i \right) = 0, \quad k = 0, \ldots, K-1$$

establishing the Hankel system.

The reduction to a linear system could also be attributed to De Prony, who had noticed that while finding α_k (or their reciprocals p_k) leads to non-linear equations, the problem becomes linear if one looks instead for the coefficients of an associated polynomial $p(D)$, where D is the derivative operator, chosen so that it annihilates the desired combination of exponentials; furthermore, De Prony's polynomial is precisely the reciprocal $s^K b_K(1/s)$ of Jacobi's polynomial.

Once the polynomial $b_K(x)$ is found, its roots, the desired cluster centers p_k, may be easily found via root-finding procedures. Finally, the weights w_k are solved from the Vandermonde system given by the first K equations.

3. The approximation procedure by mixtures of exponentials

The fact that the stationary excess moments are simply obtained from the claim moments (by the Pollaczek–Khinchine formula) suggests two possible approaches of using a mixture of exponentials approximation, either

1. for the claim distribution (to be called the "classical method")
2. or for the stationary excess distribution (to be called "Ramsay's method").

Note the two approaches are different in the numbers of moments they use, and so not immediately comparable. This suggests the following procedure:

1. Decide on a "reliable" number N of moments $m_1, m_2, \ldots, m_N$ to be estimated from the data.
2. If $N = 2K - 1$, use the classical approximation approach via a mixture of exponentials

$$\sum_{i=1}^K w_i \frac{\mathrm{e}^{\frac{-x}{p_i}}}{p_i}$$

   where p_i may be found as the roots of the polynomial given in (16) whose coefficients are obtained from (11) and w_i are obtained subsequently from the first K equations of the system (7).
3. Get the ultimate ruin approximation from (4).
4. If $N = 2K$, use "Ramsay's method": use the Pollaczek–Khinchine formula to get the first $2K-1$ moments of the stationary excess-distribution and then approximate the ultimate ruin probability by mixture of exponentials of order K, precisely as described above.

4. Non-negativity of the resulting approximation

A non-trivial problem is checking if the resulting approximation is a positive measure.

An obvious necessary (but not sufficient) condition for $\bar{F}_K(t)$ to be a bona-fide ccdf for a non-negative rv is that the exponents, α_i, have negative real parts. The approximation can result in negative weights and complex exponents, in which case the positivity of the approximating density $f_K(t) \geq 0,\ \forall t \geq 0$ for fixed K is not guaranteed (despite the convergence when $K \to \infty$).

For small values of K the positivity of the measure can be checked. For $K = 2$, which involves m_1, m_2 and m_3, we have the following bounds [19]

$$0 < m_1 < \infty$$
$$\frac{1}{2} < cv < \infty$$
$$\begin{cases} 3m_1^3(3cv - 1 + \sqrt{2}(1-cv)^{3/2}) < m_3 \leq 6m_1^3 cv & \text{if } \frac{1}{2} < cv \leq 1 \\ \frac{3}{2}m_1^3(1+cv)^2 \leq m_3 < \infty & \text{if } 1 < cv \end{cases}$$

where $cv = 1/2$ and $m_3 = 3m_1^3(3cv - 1 + \sqrt{2}(1-cv)^{3/2})$ are excluded because they require a multiple pole. For $K = 2$ the positivity of the approximating measure can be checked based on the moments themselves. For $K = 3$, which involves $m_1, \ldots, m_5$, even if no explicit expressions for the bounds of the moments are available, the positivity of the approximating pdf can be checked based on the necessary and sufficient conditions given in [20].

For $K > 3$ only sufficient conditions of the positivity of the pdf are known [21], and they apply only to subclasses of the family of densities considered in this paper. In particular, no general results are known in case of complex poles. In practice, this may require "user rejection or adjusting" of approximations (see [22]).

In the following section we provide several numerical examples testing the positivity of the resulting approximation in each case.

5. Numerical results

In this section, we illustrate numerically the relative accuracy obtained by the method of exponential mixtures presented in this paper. We provide below:

1. Comparisons of our approximations for different values of K, denoted by $PT_{K=j}$, to the exact values of the ruin probability $\psi(u)$ (when known), and to some other previous approximations. Notably the "exponential DeVylder approximation" (see [23]) denoted by DV, the Badescu and Standford approximation (see [24]) denoted by BS, and the "four moments Gamma approximation" (4MG) of [25]. With the exception of the log-normal distribution, for which all methods perform poorly, it is shown that the moments approximation performs better.
2. Relative errors ϵ_A, which are given for any approximation $\psi_A(u)$ of $\psi(u)$ by :
$$\epsilon_A(u) = \frac{\psi_A(u) - \psi(u)}{\psi(u)}.$$
We compute also two aggregate quantities over the different values of u: the average relative error $\bar{\epsilon}$ (the mean of the different relative errors absolute values) and the standard deviation mean of those relative errors σ_ε.
3. We also give the nodes and the weights for the exponential mixture approximations of the claims distribution (the support points p_i, denoted by the vector $\mathbf{p}$, and the masses w_i, denoted by the vector $\mathbf{w}$) and the exponents and coefficients of the exponential mixtures approximations of the ultimate ruin probability (the vector $\mathbf{r}$ contains the exponents while their coefficients are given by the vector $\mathbf{C}$).

If not mentioned, the claim arrivals rate is taken to be $\lambda = 1$. In most of the case, we use the loading factor θ instead of c in input data, recall that $c = (1+\theta)\lambda m_1$.

Remark 1. The 4MG method, like DeVylder's, replaces the original process by one with Gamma(α, β) claims, and artificial parameters $\tilde{\lambda}, \tilde{\theta}$, chosen such that the first four moments of the original process (computed using the data and the real λ, θ) equal those of the approximating Gamma process. This results in estimated parameters, like $\hat{\alpha} = \tilde{CV}^{-1}$, where $\tilde{CV}$ is the approximating coefficient of variation. Then, the ruin probabilities of the approximating process are computed using Thorin's integral formula (17) — see [26], which is valid when the shape parameter is smaller than 1 (completely monotone claims). However, we would like to note that the use of Thorin's formula is not essential, since rational approximation + partial fractions work also very well in this case (for example, the classic Padé approximation of Mathematica, or the one implemented here, as well as the rational approximations proposed in [27] and implemented in [28]).

On the other hand, Burnecki, Mista, and Weron have not proposed a solution for the case when the shape parameter α is greater than 1, and so their method still requires further clarification.

We believe that the procedure given in Example 4 could provide an efficient solution for this case.

Before our objects of study, the Gamma and Log-Normal, we start with a "training section" in which the claims distribution is a mixture of exponentials.

5.1. Mixed exponential distributed claims

This case reduces of course to polynomial root-solving and partial fractions, included automatically in Mathematica's command InverseLaplaceTransform. We implemented it however also by the method of moments, as a testing case.

Table 1
Ruin probabilities and approximations when the claims amount distribution is a mixture of five exponentials with pdf $b(y) = \frac{315e^{-5y}}{128} + \frac{7e^{-4y}}{8} + \frac{27e^{-3y}}{64} + \frac{3e^{-2y}}{16} + \frac{7e^{-y}}{128}$ and $c = 2/5$.

u	$\psi(u)$	DV	$4MG$	BS
0.5	0.52760668	0.51983648	0.52012656	0.5295251
1	0.3907689	0.40024733	0.39731662	0.39140827
1.5	0.29644094	0.30816985	0.3053384	0.29621049
2	0.22751173	0.23727492	0.2351673	0.2271435
2.5	0.17569297	0.18268948	0.18129702	0.17543635
3	0.13614869	0.1406615	0.13983236	0.13602249
3.5	0.10571757	0.10830211	0.10787726	0.10567941
4	0.082185905	0.083387053	0.083235577	0.082194001
4.5	0.063937514	0.064203739	0.064227316	0.063964344
5	0.049762171	0.049433574	0.049561983	0.049792801
u	$PT_{K=2}$	$PT_{K=3}$	$PT_{K=4}$	$PT_{K=5}$
0.5	0.52558109	0.52757065	0.52760712	0.52760668
1	0.38929825	0.39081373	0.39076928	0.3907689
1.5	0.29642303	0.29647712	0.29644048	0.29644094
2	0.22838235	0.22751417	0.22751148	0.22751173
2.5	0.1768237	0.17567869	0.17569302	0.17569297
3	0.13718031	0.13613334	0.13614882	0.13614869
3.5	0.10651233	0.10570724	0.10571765	0.10571757
4	0.082728172	0.08218091	0.082185929	0.082185905
4.5	0.064263778	0.063936237	0.063937509	0.063937514
5	0.049923293	0.049762925	0.049762156	0.049762171

Table 2
Relative errors (expressed in %) of the different approximations when the claims amount distribution is a mixture of five exponentials with pdf $b(y) = \frac{315e^{-5y}}{128} + \frac{7e^{-4y}}{8} + \frac{27e^{-3y}}{64} + \frac{3e^{-2y}}{16} + \frac{7e^{-y}}{128}$ and $c = 2/5$.

u	ε_{DV}	ε_{4MG}	ε_{BS}	$\varepsilon_{PT_{K=2}}$	$\varepsilon_{PT_{K=3}}$	$\varepsilon_{PT_{K=4}}$	$\varepsilon_{PT_{K=5}}$
0.5	−1.4727256	−1.4177443	0.36360879	−0.38392051	−0.00682921	0.00008479	0.00000000
1	2.4255844	1.6755978	0.16361683	−0.37634927	0.01147098	0.00009480	0.00000000
1.5	3.9565739	3.0014267	−0.077739939	−0.00604340	0.0122032	−0.00015673	0.00000000
2	4.2912935	3.3649155	−0.161849	0.38267189	0.00107315	−0.00010793	0.00000000
2.5	3.9822381	3.1896857	−0.14605933	0.64358741	−0.00812708	0.00003245	0.00000000
3	3.3146147	2.7056212	−0.092691612	0.757714	−0.01127483	0.00009192	0.00000000
3.5	2.4447583	2.0428853	−0.036097049	0.75176907	−0.00977254	0.00007332	0.00000000
4	1.4615002	1.2771916	0.0098499861	0.65980506	−0.00607771	0.00002835	0.00000000
4.5	0.41638208	0.45325804	0.041961713	0.51028575	−0.00199780	-9.149×10^{-6}	0.00000000
5	−0.66033324	−0.40228962	0.061553077	0.32378406	0.00151577	−0.00002913	0.00000000
$\bar{\varepsilon}$	2.4426004	1.9530616	0.11550273	0.47959304	0.00703423	0.00007087	0.00000000
σ_{ε}	1.3426176	1.0329913	0.097306862	0.22124094	0.0040668084	0.00004336	0.00000000

Example 1 (*Cramér–Lundberg Model with Exponential Mixtures Jumps of Order Five*)**.** The first example is produced by the method of "rational ruin probabilities" of [29]. Suppose X is a Cramér–Lundberg process with cumulant generating function:

$$\kappa(s) = \frac{2s}{5} + \frac{7}{128(s+1)} + \frac{3}{16(s+2)} + \frac{27}{64(s+3)} + \frac{7}{8(s+4)} + \frac{315}{128(s+5)} - 1$$

corresponding to $c = 2/5$, the claim density is:

$$b(y) = \frac{315e^{-5y}}{128} + \frac{7e^{-4y}}{8} + \frac{27e^{-3y}}{64} + \frac{3e^{-2y}}{16} + \frac{7e^{-y}}{128},$$

and $\lambda = 1$.

This yields the ruin probability

$$\psi(y) = \frac{245e^{-9y/2}}{32768} + \frac{135e^{-7y/2}}{8192} + \frac{567e^{-5y/2}}{16384} + \frac{735e^{-3y/2}}{8192} + \frac{19845e^{-y/2}}{32768}.$$

The results are presented in Table 1 for the different ruin probabilities, Table 2 for the relative errors and Table 3 for the nodes and weights obtained. All the resulting approximating mixtures of exponentials satisfy the sufficient condition presented in [21] and hence provide valid distributions.

Table 3
Nodes and weights **p**, **w**, **r** and **C**, when the claims amount distribution is a mixture of five exponentials with pdf $b(y) = \frac{315e^{-5y}}{128} + \frac{7e^{-4y}}{8} + \frac{27e^{-3y}}{64} + \frac{3e^{-2y}}{16} + \frac{7e^{-y}}{128}$ and $c = 2/5$.

K	2	3
p	{0.239472, 0.884091}	{0.996179, 0.214587, 0.431653}
w	{0.903679, 0.0963212}	{0.0563306, 0.745812, 0.197857}
r	{0.478582, 2.27756}	{0.0208753, 0.407411, 0.986318}
C	{0.646656, 0.122575}	{0.90565, 0.00274443, 0.000696077}
K	4	5
p	{0.203974, 0.302201, 0.495591, 0.999975}	$\left\{\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}$
w	{0.594914, 0.248887, 0.101497, 0.054702}	$\left\{\frac{63}{128}, \frac{7}{32}, \frac{9}{64}, \frac{3}{32}, \frac{7}{128}\right\}$
r	{0.0206956, 0.280746, 0.482593, 0.990417}	$\left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}\right\}$
C	{0.904861, 0.0022125, 0.00134478, 0.000672364}	$\left\{\frac{19845}{32768}, \frac{735}{8192}, \frac{567}{16384}, \frac{135}{8192}, \frac{245}{32768}\right\}$

Note.

(a) As expected, $PT_{K=5}$ reproduces the correct support points and weights, yielding consequently the exact values of the ruin probability.

(b) DV is the worst approximation, after 4MG, BS and $PT_{K=2}$. The performance of BS is remarkable, which leads us to conjecture that this is asymptotically correct to second order ([30] has shown that DeVylder is asymptotically correct to first order). Note however that since BS uses a wrong loading factor, it is incorrect for small values of u, and that if large values of u only are of interest, it is easy to modify the $PT_{K=3}$, $PT_{K=4}$, etc. to DeVylder type approximations (just by applying the moments to the process rather than to the claims, precisely as BS have done to obtain a "second order DeVylder type approximation").

In this example, from $K = 3$ the accuracy obtained by the method of moments is far and away better than the other approximations. For $K = 3$ we obtain an average relative error of $\bar{\varepsilon} = 0.007\%$ which is better compared to BS with $\bar{\varepsilon} = 0.11\%$. The approximation with the method of moments is quasi-exact for $K = 4$ (at least ups to 6 digits), and $K = 5$ gives the exact values.

(c) For a value of K which is greater than the order of the initial mixed exponential distribution (here for $K > 5$), numerical errors take over due to singular moment matrices, and we obtain inconsistent values, like exponents greater than zero (hence incorrect for large u), and negative ruin probabilities. For $K = 6$ for example, the approximation contains components with positive exponentials (incorrect for large u). In a more elaborate implementation, these should be of course removed, by imposing a negativity condition on the Cramér Lundberg roots. Effectively, that would impose using the correct K when K is bigger than that. This point is of course important when the "correct K" is not known.

5.2. Gamma processes

Let us consider now "Gamma processes" with cumulant generating function

$$\kappa(s) = cs + \lambda(b^*(s) - 1),$$

where

$$b^*(s) = \int_0^\infty e^{-sx} b_{\alpha,\beta}(x)dx = (1 + s\beta)^{-\alpha},$$

with $b_{\alpha,\beta}(x) = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\beta^\alpha \Gamma(\alpha)}dx$, and $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1}e^{-u}du$. The survival probability transform is: $\overline{\psi}^*(s) = \frac{\alpha\beta(1-\rho)}{\alpha\beta s^2 - s\rho(1-(1+s\beta)^{-\alpha})}$.

Notes. (1) For $\alpha \in \mathbb{N}$, the distribution (also called Erlang) has rational Laplace transform. For other α, any rational approximation for $(1+z)^{-\alpha}$, like for example Gauss's continued fraction representation for the binomial series, will provide one for ruin probabilities.

For example, when $\alpha = 1/2$, $\beta = 1$, $\rho = 3/4$, $\lambda m_1 = 1$, we get

$$\overline{\psi}^*(s) = \frac{5/2}{4s - 3 + 3(1+s)^{-1/2}},$$

the Padé(0, 1) approximation is $27/(77s + 60)$ and the approximate ruin probability is $(27/77)\exp(-60/77y)$.

(2) In the case $\alpha \in (0, 1)$, [31] showed that the Gamma distribution is completely monotonic, and the resulting ruin probability is given by:

$$\psi(u) = \frac{(c-1)(1-\alpha^{-1}\gamma)}{1 - c\gamma - c(1-\alpha^{-1}\gamma)} e^{-\gamma u} + \frac{(c-1)}{\pi}\sin(\alpha\pi)\int_\alpha^\infty \frac{(\alpha^{-1}x - 1)^{-\alpha}e^{-xu}dx}{\zeta_1(x)} \tag{17}$$

where γ is the adjustment coefficient and $\zeta_1(x) = [1 + cx - (\alpha^{-1}x - 1)^{-\alpha}\cos(\alpha\pi)]^2 + \sin^2(\alpha\pi)(\alpha^{-1}x - 1)^{-2\alpha}$.

Table 4
Ruin probabilities and approximations when the claims amount distribution is Gamma with parameters $\alpha = \beta = 0.01$.

u	$\psi(u)$	DV	$4MG$	BS
0	0.90909091	0.88286713	0.90909091	0.89913827
300	0.52114308	0.52253878	0.52114308	0.52107431
600	0.30866782	0.30927278	0.30866782	0.30866876
900	0.18286631	0.18304795	0.18286631	0.18286826
1200	0.10833788	0.1083398	0.10833788	0.10833887
1500	0.064184065	0.064122614	0.064184065	0.064184512
1800	0.038025428	0.037951976	0.038025428	0.03802561
2100	0.022527915	0.022462473	0.022527915	0.022527975
2400	0.013346515	0.013294767	0.013346515	0.013346522
2700	0.0079070552	0.0078687168	0.0079070552	0.0079070421
3000	0.0046844829	0.0046572236	0.0046844829	0.0046844651
u	$PT_{K=2}$	$PT_{K=3}$	$PT_{K=4}$	$PT_{K=5}$
0	0.90909091	0.90909091	0.90909091	0.90909091
300	0.5225258	0.52107463	0.52115121	0.52114176
600	0.30926783	0.30866874	0.30866735	0.30866788
900	0.18304664	0.18286825	0.18286627	0.18286631
1200	0.10833998	0.10833886	0.10833789	0.10833788
1500	0.064123285	0.064184509	0.064184068	0.064184065
1800	0.037952708	0.038025609	0.038025429	0.038025428
2100	0.022463104	0.022527974	0.022527915	0.022527915
2400	0.013295258	0.013346522	0.013346515	0.013346515
2700	0.0078690767	0.0079070422	0.0079070553	0.0079070552
3000	0.0046574777	0.0046844652	0.0046844829	0.0046844829

Table 5
Relative errors (expressed in %) of the different approximations when the claims amount distribution is Gamma with parameters $\alpha = \beta = 0.01$.

u	ε_{DV}	ε_{4MG}	ε_{BS}	$\varepsilon_{PT_{K=2}}$	$\varepsilon_{PT_{K=3}}$	$\varepsilon_{PT_{K=4}}$	$\varepsilon_{PT_{K=5}}$
0	−2.8846154	0.00000000	−1.0947901	0.00000000	0.00000000	0.00000000	0.00000000
300	0.26781356	0.00000000	−0.0131969	0.26532423	−0.01313592	0.00155906	−0.00025367
600	0.19598924	0.00000000	0.00030297	0.19438553	0.0002961	−0.00015353	0.0000192
900	0.0993301	0.00000000	0.00106807	0.09861095	0.00106156	−0.00002244	-1.1×10^{-6}
1200	0.00177217	0.00000000	0.00090743	0.00193586	0.00090221	1.64×10^{-6}	-4.84×10^{-7}
1500	−0.09574235	0.00000000	0.00069545	−0.09469752	0.00069152	3.41×10^{-6}	-5.47×10^{-8}
1800	−0.19316482	0.00000000	0.00048041	−0.19124058	0.00047779	3.10×10^{-6}	4.28×10^{-9}
2100	−0.29049249	0.00000000	0.00026519	−0.28769052	0.00026386	2.63×10^{-6}	9.04×10^{-9}
2400	−0.38772525	0.00000000	0.00004995	−0.38404728	0.00004992	2.14×10^{-6}	8.45×10^{-9}
2700	−0.48486321	0.00000000	−0.00016529	−0.48031093	−0.00016403	1.66×10^{-6}	7.67×10^{-9}
3000	−0.58190643	0.00000000	−0.00038053	−0.57648155	−0.00037797	1.18×10^{-6}	6.22×10^{-9}
$\bar{\varepsilon}$	0.49015086	0.00000000	0.11119218	0.19982434	0.00170429	0.00017496	0.00002745
σ_{ε}	0.8096185	0.00000000	0.327888	0.15003673	0.00382545	0.00046354	0.00007562

Table 6
Nodes and weights **p**, **w**, **r** and **C**, when the claims amount distribution is Gamma with parameters $\alpha = \beta = 0.01$.

K	2	3
p	{0.250935, 67.0824}	{0.111604, 35.7971, 84.6913}
w	{0.988792, 0.0112083}	{0.981263, 0.0142428, 0.0044944}
r	{0.00424887, 3.22652}	{0.0041959, 0.01980, 8.20676}
C	{0.715155, 0.0540757}	{0.695087, 0.0529967, 0.021147}
K	4	5
p	{0.0627924, 21.4232, 59.2601, 91.254}	{0.0401917, 14.1048, 41.8085, 72.4618, 94.3624}
w	{0.975791, 0.0151844, 0.00656698, 0.00245766}	{0.971518, 0.0155776, 0.0074348, 0.00391286, 0.00155637}
r	{0.00419508, 0.0132962, 0.037476, 15.1758}	{0.00419507, 0.0116723, 0.0197509, 0.0610077, 24.1341}
C	{0.694357, 0.0231147, 0.0403739, 0.0113853}	{0.694337, 0.0113469, 0.0269749, 0.0294398, 0.00713266}

Example 2 (*Gamma Distributed Claims with* $\alpha = \beta = 0.01$). This example appears frequently in the literature (see [32,30,24]) and it comes from [33], who calculate the probabilities of ruin when $\lambda = 1$ (or $\lambda = \alpha\beta = 0.0001$ under our parametrisation), and with $\theta = 0.1$, via Thorin's formula (17) (since $\alpha < 1$, we are in the completely monotone case).

The moments are given by $m_k = \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)}\beta^{-k}$. The results are presented in Tables 4–6. In this case, as the measure to be approximated is completely monotone, the resulting approximations are valid distributions.

Table 7
Ruin probabilities and approximations when the claims amount distribution is Gamma with parameters $\alpha = 5/2$, $\beta = 1$ and $c = \frac{4}{5}(-1+4\sqrt{2})$.

u	$\psi(u)$	DV	$4MG$	BS
0.5	0.2285401	0.23724138	0.29549605	0.22880626
1	0.1896784	0.18784557	0.22550076	0.19000888
1.5	0.1544410	0.14873442	0.17340829	0.15449728
2	0.1240365	0.11776657	0.13397445	0.12391485
2.5	0.0986588	0.093246498	0.10380835	0.098481604
3	0.0779451	0.073831731	0.080580678	0.077782758
3.5	0.0612928	0.058459295	0.062622163	0.061171178
4	0.0480435	0.046287539	0.048701578	0.047963703
4.5	0.0375759	0.036650054	0.037893273	0.037529254
5	0.0293456	0.02901918	0.029492606	0.029321634
u	$PT_{K=2}$	$PT_{K=3}$	$PT_{K=4}$	$PT_{K=5}$
0.5	0.2281257128	0.2285214985	0.2285401184	0.2285406308
1	0.1890689462	0.1896828576	0.1896808091	0.1896784446
1.5	0.1540155762	0.1544593784	0.1544410893	0.1544407456
2	0.1239260605	0.1240511485	0.1240353667	0.1240365792
2.5	0.09882161784	0.098663498	0.09865791298	0.09865898729
3	0.07827630875	0.07794189751	0.077944859	0.07794519749
3.5	0.06168944457	0.06128608167	0.06129307114	0.06129286865
4	0.04843002313	0.04803660299	0.04804391089	0.04804352506
4.5	0.03790785839	0.03757057966	0.0375762479	0.03757591365
5	0.02960373393	0.0293423273	0.02934581341	0.02934561425

Table 8
Relative errors (expressed in %) of the different approximations when the claims amount distribution is Gamma with parameters $\alpha = 5/2$, $\beta = 1$ and $c = \frac{4}{5}(-1+4\sqrt{2})$.

u	ε_{DV}	ε_{4MG}	ε_{BS}	$\varepsilon_{PT_{K=2}}$	$\varepsilon_{PT_{K=3}}$	$\varepsilon_{PT_{K=4}}$	$\varepsilon_{PT_{K=5}}$
0.5	3.7940648	29.280716	0.10366572	0.1813504911	0.0081705528	0.0000232506	−0.0002009599
1	−0.96554991	18.886719	0.17496895	0.3213608596	−0.0022981180	−0.0012181020	0.0000284568
1.5	−3.6886485	12.288646	0.043025189	0.2754735509	−0.011886735	−0.0000445991	0.0001779015
2	−5.0507742	8.0168253	−0.09372384	0.08911644458	−0.0117311776	0.0009922731	0.0000147455
2.5	−5.4856719	5.2197853	−0.17938595	−0.1649519265	−0.0046826881	0.0009782496	−0.0001106544
3	−5.2797839	3.378576	−0.2109318	−0.4249032429	0.0041309871	0.0003315291	−0.0001027299
3.5	−4.6264521	2.1650686	−0.20214385	−0.6470456522	0.0110456083	−0.0003577912	−0.0000274255
4	−3.6580434	1.3664892	−0.16931027	−0.8044331569	0.0144492539	−0.0007617438	0.0000413533
4.5	−2.4658239	0.84266714	−0.12607053	−0.8833259164	0.0142663370	−0.0008184345	0.0000710975
5	−1.1127946	0.50047711	−0.082136053	−0.879521201	0.0112642504	−0.0006152165	0.0000634464
$\bar{\varepsilon}$	3.6127607	8.194597	0.138536	0.467148	0.0093925708	0.0006141189	0.0000838771
σ_ε	1.5490615	8.9666656	0.053770	0.294120	0.004110245	0.0003902519	0.0000608669

Note. As 4MG uses numerical integration of Thorin's formula (17) for Gamma claims with the approximated parameters, it may be tuned to become virtually exact in this case.

The exponential mixture approximations are quasi-exact from $K = 3$, at least up to 5 digits. The average relative error falls to $\bar{\varepsilon} = 0.001\%$ for $K = 3$ and $\bar{\varepsilon} = 0.0001\%$ for $K = 4$. For $K = 5$ we can assert that the approximation yields exact values (up to 9 or 10 digits for the majority of the values). In fact, since we are comparing here the results of the numerical integration (17), it is not clear which of the two works better. To the best of our knowledge, such an accuracy has not yet been obtained in previous related work.

Example 3 (*Gamma Distributed Claims with $\alpha = 5/2$ and $\beta = 1$*)**.** For a second example, let us take Gamma(5/2, 1) claims (since $\alpha > 1$, we are **not in the completely monotone case**), and $c = \frac{4}{5}(-1+4\sqrt{2})$, which ensures $\gamma = 1/2$, implying

$$\psi^*(s) = \frac{5s(s+1)^{5/2} - 2(s+1)^{5/2} + 2}{\zeta_2(x)}$$

with $\zeta_2(x) = 2s(8\sqrt{2}s(s+1)^{5/2} - 2s(s+1)^{5/2} - (s+1)^{5/2} + 1)$.

For this example, we do not have either the exact result, or a previous authoritative study. As a proxy for an exact result, we used the results obtained from numerical inverse Laplace by Weeks' method as provided in [34,35]. This method provides also an estimate of the error of the obtained values which in our case was always less than 10^{-13}.

The approximation results are presented in Tables 7–9.

Here, the accuracy of the method of moments is not so sharp as in the completely monotone case, and the value of K does not strongly influence the accuracy. Despite that, the exponential mixture approximation is better than all the

Table 9
Nodes and weights **p**, **w**, **r** and **C**, when the claims amount distribution is Gamma with parameters $\alpha = 5/2$, $\beta = 1$ and $c = \frac{4}{5}(-1 + 4\sqrt{2})$.

K	2	3
p	{1.16667 − 0.311805i, 1.16667 + 0.311805i}	{0.576576, 1.06171 − 0.104653i, 1.06171 + 0.104653i}
w	{0.5 + 2.13809i, 0.5 − 2.13809i}	{0.830764, 0.0846178 + 8.79727i, 0.0846178 − 8.79727i}
r	{0.511202, 0.981475}	{0.576576, 1.06171, 1.06171}
C	{0.392984, −0.124676}	{0.359472, −0.224084, 0.132919}
K	4	5
p	{0.317198, 0.759302,1.03318−0.0545318i, 1.03318 + 0.0545318i}	{0.202246, 0.522173, 0.844953, 1.02087−0.0338508i, 1.02087 + 0.0338508i}
w	{0.0613626, 3.17895, −1.12016 + 21.8349i, −1.12016 − 21.8349i}	{0.0140594, 0.260825, 7.45863, −3.36676 + 43.3197i, −3.36676 − 43.3197i}
r	{0.500106, 1.32319 − 0.230044i, 1.32319 + 0.230044i, 3.14619}	{0.500105, 1.28157 − 0.207182i, 1.28157 + 0.207182i, 1.88645, 4.94299}
C	{0.359075, −0.0461114 + 0.0356539i, −0.0461114 − 0.0356539i, 0.00145442}	{0.359065, −0.0511728 + 0.0183542i, −0.0511728 − 0.0183542i, 0.0113732, 0.000214386}

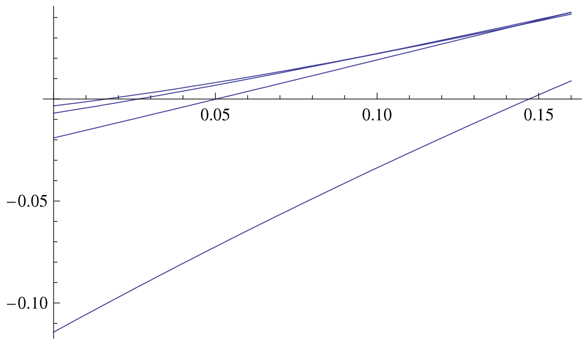

Fig. 1. Pdf associated with the approximating mixture of exponentials around 0 when the claims amount distribution is Gamma with parameters $\alpha = 5/2$, $\beta = 1$. As K is increased the pdf is less negative.

Table 10
Nodes and weights **p**, **w** when the claims amount distribution is Gamma with parameters $\alpha = 1/2$, $\beta = 1$.

K	2	3
p	{0.1464466, 0.853553}	{0.066987, 0.5, 0.933013}
w	{1/2, 1/2}	{1/3, 1/3, 1/3}
K	4	5
p	{0.03806, 0.308658, 0.691342, 0.9619398}	{0.024472, 0.206107, 0.5, 0.793892, 0.975528}
w	{1/4, 1/4, 1/4, 1/4}	{1/5, 1/5, 1/5, 1/5, 1/5}

Table 11
Ruin probabilities and approximations when the claims amount distribution is Gamma with parameters $\alpha = 5/2$, $\beta = 1$ and $c = \frac{4}{5}(-1 + 4\sqrt{2})$, and the approximation is performed by convoluting the approximation of a Gamma(1/2, 1) and Gamma(2, 1).

u	$\psi(u)$	$PT_{K=2}$	$PT_{K=3}$	$PT_{K=4}$	$PT_{K=5}$
0.5	0.2285401	0.2284962256	0.2285336175	0.2285399515	0.2285404042
1	0.1896784	0.1895984397	0.1896799045	0.1896796391	0.1896784825
1.5	0.1544410	0.1543922322	0.154447576	0.1544410301	0.1544408595
2	0.1240365	0.1240375462	0.1240411664	0.1240359758	0.1240365906
2.5	0.0986588	0.0986952899	0.09865961225	0.09865844178	0.09865894378
3	0.0779451	0.0779962295	0.07794333916	0.07794504669	0.07794516123
3.5	0.0612928	0.0613431694	0.0612903228	0.06129299376	0.06129285754
4	0.0480435	0.0480848656	0.04804137621	0.04804372696	0.04804353116
4.5	0.0375759	0.0376056260	0.03757453864	0.03757607271	0.03757592492
5	0.0293456	0.0293642802	0.02934497201	0.02934569688	0.02934562338

other approximations, starting from $K = 3$. In this case the approximating mixtures of exponentials do not provide valid distributions. In particular, as it is depicted in Fig. 1, the associated pdf is negative around 0 for all the approximations. The higher the order of the approximation the less negative the associated pdf but, as it can be shown by applying results from [36] regarding relations of the moments and the derivatives of the pdf at 0, the negativity remains for any degree.

Table 12
Relative errors (expressed in %) of the different approximations when the claims amount distribution is Gamma with parameters $\alpha = 5/2, \beta = 1$ and $c = \frac{4}{5}(-1 + 4\sqrt{2})$ and the approximation is performed by convoluting the approximation of a Gamma(1/2, 1) and Gamma(2, 1).

u	$\varepsilon_{PT_{K=2}}$	$\varepsilon_{PT_{K=3}}$	$\varepsilon_{PT_{K=4}}$	$\varepsilon_{PT_{K=5}}$
0.5	0.01922898711	0.002867783548	0.00009627915876	−0.0001017943495
1	0.04220768804	−0.0007411775601	−0.0006013019536	0.0000085143267
1.5	0.03159019834	−0.004244743402	−0.000006254974	0.000104176056
2	−0.0007648503503	−0.003683494736	0.0005012176826	0.000005568617
2.5	−0.03690682912	−0.0007441167226	0.0004422626293	−0.00006655673105
3	−0.06557448681	0.002281420982	0.00009073978609	−0.00005621797428
3.5	−0.08209371146	0.004126151367	−0.0002315527382	−0.000009309721
4	−0.08600680967	0.004514055627	−0.0003788967246	0.00002865528204
4.5	−0.07900191227	0.003730391397	−0.0003521933505	0.00004111229094
5	−0.06354397049	0.002251966879	−0.0002181296221	0.00003234523629
$\bar{\varepsilon}$	0.05069194436	0.002918530222	0.00029188286	0.00004542505863
σ_{ε}	0.02744239	0.001316393	0.0001851394	0.00003452299

Table 13
Ruin probabilities and approximations when the claims amount distribution is Log-Normal with mean $\mu = -1.62$ and variance $\sigma^2 = 3.24$.

u	θ (%)	$\psi(u)$	DV	$4MG$	BS
100	0.05	0.5507400	0.43720144	**	0.4443055
100	0.1	0.3439500	0.27694243	**	0.28250945
100	0.15	0.2357300	0.2021904	**	0.20656212
100	0.2	0.1730900	0.15908799	**	0.16264931
100	0.25	0.1338400	0.13108771	**	0.13408032
100	0.3	0.1076500	0.11144913	**	0.11402491
1000	0.05	0.0419900	0.065123171	**	0.063605328
1000	0.1	0.0109900	0.020390221	**	0.01929732
1000	0.15	0.0057400	0.010312679	**	0.0095528294
1000	0.2	0.0038400	0.0064779869	**	0.0059119941
1000	0.25	0.0028800	0.0045839046	**	0.0041383836
1000	0.3	0.002300	0.0034918423	**	0.003126773
u	θ(%)	$PT_{K=2}$	$PT_{K=3}$	$PT_{K=4}$	$PT_{K=5}$
100	0.05	0.42463735	0.43217656	0.43250966	0.43252432
100	0.1	0.2628876	0.26872275	0.26898154	0.26899293
100	0.15	0.18932436	0.19384349	0.19404412	0.19405295
100	0.2	0.14765503	0.15129729	0.15145902	0.15146614
100	0.25	0.12092371	0.12395967	0.12409447	0.12410041
100	0.3	0.10234753	0.10494438	0.10505966	0.10506474
1000	0.05	0.065205036	0.063735997	0.063670119	0.063667219
1000	0.1	0.020908287	0.019824809	0.019776899	0.01977479
1000	0.15	0.010815084	0.010050582	0.010017044	0.010015568
1000	0.2	0.0069156598	0.0063398002	0.0063146727	0.0063135677
1000	0.25	0.004962126	0.004504874	0.0044849989	0.004484125
1000	0.3	0.0038217649	0.0034444798	0.0034281283	0.0034274094

Table 14
Relative errors (expressed in %) of the different approximations when the claims amount distribution is Log-Normal with mean $\mu = -1.62$ and variance $\sigma^2 = 3.24$.

u	θ (%)	ε_{DV}	ε_{4MG}	ε_{BS}	$\varepsilon_{PT_{K=2}}$	$\varepsilon_{PT_{K=3}}$	$\varepsilon_{PT_{K=4}}$	$\varepsilon_{PT_{K=5}}$
100	0.05	−20.615638	**	−19.325725	−22.896948	−21.528025	−21.467541	−21.46488
100	0.1	−19.481777	**	−17.863221	−23.568076	−21.871567	−21.796325	−21.793014
100	0.15	−14.227974	**	−12.373429	−19.685927	−17.76885	−17.683741	−17.679995
100	0.2	−8.0894405	**	−6.0319421	−14.69465	−12.590393	−12.496956	−12.492844
100	0.25	−2.0564036	**	0.17955679	−9.6505478	−7.3821915	−7.281476	−7.2770432
100	0.3	3.5291534	**	5.9218874	−4.9256612	−2.5133483	−2.4062594	−2.4015461
1000	0.05	55.092095	**	51.477324	55.28706	51.788514	51.631625	51.624717
1000	0.1	85.534316	**	75.589811	90.248285	80.38953	79.953583	79.934399
1000	0.15	79.663402	**	66.425598	88.416098	75.097251	74.512955	74.487253
1000	0.2	68.697575	**	53.958179	80.095307	65.098964	64.444602	64.415825
1000	0.25	59.163355	**	43.693874	72.296042	56.419237	55.729129	55.698785
1000	0.3	51.819231	**	35.946654	66.16369	49.759993	49.049055	49.017799
	$\bar{\varepsilon}$	38.9975	**	32.3989	45.6607	38.5173	38.2044	38.1907
	σ_{ε}	29.4983	**	24.5115	31.385	26.3866	26.1787	26.1696

Table 15
Nodes and weights **p**, **w**, **r** and **C** when the claims amount distribution is Log-Normal with mean $\mu = -1.62$ and variance $\sigma^2 = 3.24$.

K	2	3
p	{0.949446, 233.758}	{0.947096, 223.396, 89597.7}
w	{0.99978, 0.000217147}	{0.99976, 0.000237822, 1.69987×10^{-13}}
r	{0.284639, 0.00365295}	{0.287307, 0.00379630, 0.0000111609}
C	{0.621754, 0.147476}	{0.615828, 0.15340, 5.08165×10^{-8}}
K	4	5
p	{0.946989, 222.944, 85856, 4.14723×10^{7}}	{0.947415, 222.927, 85698.3, 3.97954×10^{7}, 2.09796×10^{10}}
w	{0.99976, 0.000238789, 2.01606×10^{-13}, 3.00786×10^{-25}}	{0.999763, 0.000236882, 1.32608×10^{-11}, -1.30308×10^{-19}, 1.683342×10^{-30}}
r	{0.287429, 0.0038028, 0.0000116474, 2.41124×10^{-8}}	{0.28695, 0.00380775, 0.0000116687, 2.51285×10^{-8}, 4.76651×10^{-11}}
C	{0.6155604, 0.153670, 5.77543×10^{-8}, 4.15811×10^{-17}}	{0.6166, 0.152626, 3.79185×10^{-6}, 1.72856×10^{-11}, 1.1772×10^{-19}}

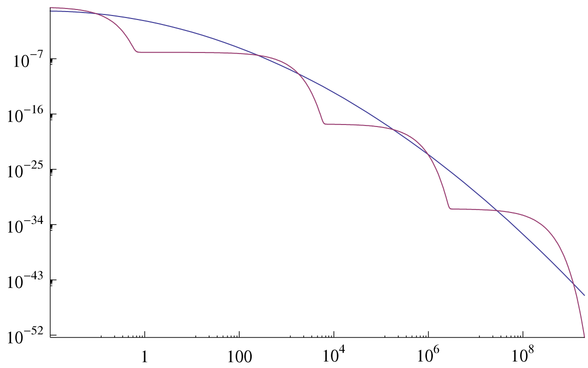

Fig. 2. Pdf associated with the approximating mixture of exponentials with $K = 4$ when the claims amount distribution is log-normal with mean $\mu = -1.62$ and variance $\sigma^2 = 3.24$ and the log-normal distribution itself.

Example 4. A possible workaround of the problem of the negativity of the approximation is to write the Gamma(5/2, 1) as the convolution of a Gamma(2, 1) distribution and a Gamma(1/2, 1) distribution. As the approximation yields valid distributions for $0 < \alpha < 1$, the positivity of the approximating distribution is guaranteed. Table 10 provides the support points (**p**) and the masses (**w**) for the mixture of exponentials approximating Gamma(1/2, 1). The distribution approximating the Gamma(5/2,1) distribution can be written in matrix exponential form by convolution of the mixture of exponentials approximating the Gamma(1/2, 1) distribution and the Gamma(2, 1) distribution (which is a matrix exponential distribution). The ruin probabilities can be then computed by (4). The accuracy is illustrated in Tables 11 and 12; note that this fix-up improves considerably the accuracy, bringing the "errors" to the same level as that obtained in the completely monotone case; for example 4×10^{-5}, when $K = 5$.

5.3. Log-Normal distributed claims

For a log-normal variable $X = e^{N_{\mu,\sigma}}$, the moments are explicit, given by: $m_k = e^{k\mu + k^2\sigma^2/2}$. Note however that the Laplace transform:

$$b^*(s) = E[e^{-sX}] = \int_0^\infty \frac{e^{-\frac{(Log(x)-\mu)^2}{2\sigma^2}}}{x\sigma\sqrt{2\pi}} e^{-sx} dx,$$

is not explicit, which makes moments based methods even more attractive.

This example was first analyzed in [26]. They assume that the claims size are log-normally distributed with mean $\mu = -1.62$ and variance $\sigma^2 = 3.24$. The approximation 4MG is not applicable in this case.

In this case as well, the sufficient condition of [21] guarantees that the resulting approximating mixture of exponentials is a valid measure. The results are reported in Tables 13–15.

As confirmed by the tables, the log-normal distribution, which is moment indeterminate, is a weak spot of moments based approximations, and ours does not depart from this rule. As we can see, here all the approximations perform badly, with BS being the champion, and increasing the value of K does not change much the average relative errors. Clearly, further research is necessary in this case. For what concerns our method, the failure may be due to the "heavy tail" of the considered log-normal distribution. Fig. 2 depicts how the approximating mixture of exponentials is waving around the log-normal distribution for $K = 5$; the apparent changes of slope on this figure are an artifact of rapid (but smooth) changes of sign in the second derivative.

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